E_8-PLUMBINGS AND EXOTIC CONTACT STRUCTURES ON SPHERES

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1. Introduction

The standard contact structure ξ_{st} on the unit sphere $S^{2n-1} = \partial D^{2n} \subset \mathbb{C}^n$ can be defined as the hyperplane field of complex tangencies. In other words, if we write J_0 for the complex structure on (the tangent bundle of) $\mathbb{C}^n$, then $\xi_{st}(p) = T_pS^{2n-1} \cap J_0(T_pS^{2n-1})$ for all $p \in S^{2n-1}$.

Conversely, given any contact structure $\xi = \ker \alpha$ on S^{2n-1} (with α a 1-form such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form defining the standard orientation), there is a homotopically unique complex bundle structure J on ξ such that $d\alpha(., J.)$ is a J-invariant Riemannian metric on ξ. This extends to a complex structure J on $T\mathbb{R}^{2n}|_{S^{2n-1}}$ by requiring that J send the outer normal of S^{2n-1} to a vector field X on S^{2n-1} with $\alpha(X) > 0$. If this J extends as an almost complex structure over the disc D^{2n}, then ξ is called *homotopically trivial*.

The complex structure $J|_\xi$ and the trivial line bundle spanned by the vector field X can be interpreted as a reduction of the structure group of TS^{2n-1} to $\mathrm{U}_{n-1} \times 1$. Such a reduction is called an *almost contact structure*. Homotopy classes of almost contact structures on S^{2n-1} are classified by

$$\pi_{2n-1}(\mathrm{SO}_{2n-1}/\mathrm{U}_{n-1}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n \equiv 0 \bmod 4, \\ \mathbb{Z}_{(n-1)!} & \text{for } n \equiv 1 \bmod 4, \\ \mathbb{Z} & \text{for } n \equiv 2 \bmod 4, \\ \mathbb{Z}_{(n-1)!/2} & \text{for } n \equiv 3 \bmod 4. \end{cases}$$

Homotopically trivial contact structures are those that induce the same almost contact structure as ξ_{st} and correspond to the trivial element in the respective group above.

The purpose of this note is to prove the following result, where we call a contact structure on S^{2n-1} *exotic* if it is not diffeomorphic to ξ_{st}.

Theorem 1. *In any odd dimension $2n - 1 \geq 3$, the standard sphere S^{2n-1} admits exotic but homotopically trivial contact structures.*

For the 3-sphere this was proved by Bennequin [2]. The (by [6] essentially unique) exotic but homotopically trivial contact structure on S^3 is not symplectically fillable.

In higher dimensions, by contrast, exoticity of (S^{2n-1},ξ) can be shown by exhibiting a symplectic filling of (S^{2n-1},ξ) that does not contain symplectic 2-spheres and is not diffeomorphic to a disc D^{2n}. This result is due to Eliashberg, Floer, Gromov and McDuff, see [7].

In that paper, Eliashberg used this result to derive Theorem 1 for spheres of dimension $4k+1$. Notice that this corresponds to $n = 2k+1$ odd, in which case there are only finitely many homotopy classes of almost contact structures. For that reason, it is enough to find *some* exotic (S^{2n-1},ξ) (with a symplectic filling as described) — Eliashberg did this by using a plumbing construction. An exotic but homotopically trivial contact structure on S^{2n-1} can then be constructed simply by taking the connected sum of suitably many copies of (S^{2n-1},ξ): the result of [7] applies to the boundary connected sum of the fillings.

For n even, this approach fails. In [8] a more careful study of the homotopy classes of almost contact structures was undertaken, and Theorem 1 was proved for S^7 and spheres of dimension $\equiv 3 \bmod 8$. The general case of spheres of dimension $\equiv 7 \bmod 8$ (that is, $n \equiv 0 \bmod 4$), however, had remained elusive. In the present note, we show that these remaining cases can be settled by extending ideas from [7] and [8]. Starting from an E_8-plumbing, we show how to keep control of the homotopical information necessary to prove Theorem 1. In the process, we construct almost complex manifolds that are of some independent interest.

The 'classical' methods of the present note do not allow us to distinguish different exotic contact structures in the same homotopy class of almost contact structures. For that, contact homological methods are required, see [22], [3] (again, those papers only deal with spheres of dimension $4k+1$).

For general homotopy classes we have the following statement.

Theorem 2. *The standard spheres of dimension* $\equiv 1,3,5 \bmod 8$ *(and greater than* 1*) admit exotic contact structures in every homotopy class of almost contact structures; so does* S^7*. The spheres of dimension* $\equiv 7 \bmod 8$ *admit exotic contact structures in every stably trivial homotopy class of almost contact structures.*

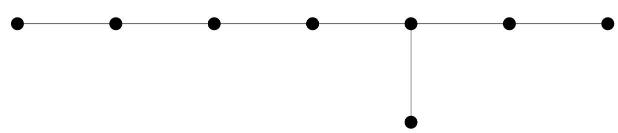
For dimension 3 this is due to Lutz [16]; exoticity follows from these structures being overtwisted. For dimensions $2n-1 \equiv 1,5 \bmod 8$ (and implicitly for $2n-1 = 7$ or $\equiv 3 \bmod 8$) this was proved in [8]. The exoticity of the structures follows from the same fillability argument as above. (A

contact structure in a nontrivial homotopy class of almost contact structures need not be exotic, *a priori*: the diffeomorphism group of S^{2n-1} may act nontrivially on these homotopy classes.)

We do not know anything about the realisation of stably nontrivial homotopy classes of almost contact structures on spheres of dimension $\equiv 7$ mod 8 (and greater than 7). At the end of this note we show that, at least on S^{15}, such homotopy classes cannot be realised by contact structures that are symplectically fillable by some highly connected manifold. In other words, the construction used to prove Theorems 1 and 2 does not allow one to extend Theorem 2 to all homotopy classes of almost contact structures.

2. The E_8-plumbing

Let DTS^{2m} be the tangent unit disc bundle of the $2m$-dimensional sphere, $m \geq 2$. It is well-known that the plumbing of eight copies of DTS^{2m} according to the E_8-graph



yields a manifold V^{4m} with boundary $\partial V^{4m} = \Sigma^{4m-1}$ a homotopy sphere, cf. [4], [13, §8]. The intersection matrix of V^{4m} (with the natural orientation of that manifold) is the E_8-matrix

$$\begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{pmatrix}$$

(with zeros in the blank spaces). In fact, Σ^{4m-1} is a generator of the group bP_{4m} of homotopy $(4m-1)$-spheres bounding a parallelisable manifold, see [15] and [13, §10]. The order $|bP_{4m}|$ of this group is given by

$$|bP_{4m}| = 2^{2m-2}(2^{2m-1} - 1) \cdot \text{numerator}\left(\frac{4B_m}{m}\right),$$

where B_m denotes the mth Bernoulli number. (In [15] this was only proved up to a potential factor of 2 in the case m even; this was later settled by

Mahowald [17], cf. [12, p. 771].) The denominator of B_m is square-free and divisible by 2 [19, p. 284]. This implies

$$|bP_{4m}| = 2^{2m-2}(2^{2m-1}-1)\cdot a_m \cdot \text{numerator}\left(\frac{B_m}{4m}\right),$$

where $a_m = 2$ for m odd, $a_m = 1$ for m even.

The boundary connected sum of $|bP_{4m}|$ copies of V^{4m} has boundary S^{4m-1}. Write W_0^{4m} for the closed, orientable $4m$-manifold obtained by attaching a $4m$-disc D^{4m} along the boundary. (There is a choice here, since for some m there are diffeomorphisms of S^{4m-1} that do not extend over D^{4m}, but the resulting ambiguity is of no consequence for our considerations.)

We collect some well-known topological information about W_0^{4m}, cf. [12, p. 770]. The manifold W_0^{4m} is $(2m-1)$-connected and its middle cohomology group is $H^{2m}(W_0^{4m}) \cong \mathbb{Z}^{\oplus 8|bP_{4m}|}$. The plumbing V^{4m} has as deformation retract a union of eight $2m$-spheres intersecting one another according to the E_8-graph. The restriction of the tangent bundle TV^{4m} to each of these spheres is isomorphic to the Whitney sum $TS^{2m}\oplus TS^{2m}$, which is stably trivial. It follows that TV^{4m} is trivial, and so is $T(W_0^{4m} - D^{4m})$. Hence, for m even, the Pontrjagin class $p_{m/2}(W_0^{4m})$ is trivial.

Lemma 3. *The Pontrjagin number $p_m[W_0^{4m}]$ is given by*

$$p_m[W_0^{4m}] = a_m \cdot \text{denominator}\left(\frac{B_m}{4m}\right)\cdot (2m-1)!,$$

where $a_m = 2$ for m odd, $a_m = 1$ for m even.

Proof. The signature of the E_8-matrix is equal to 8, so the signature of W_0^{4m} equals

$$\sigma(W_0^{4m}) = 8|bP_{4m}| = 2^{2m+1}(2^{2m-1}-1)\cdot a_m \cdot \text{numerator}\left(\frac{B_m}{4m}\right). \tag{1}$$

The coefficient of p_m in the L-polynomial $L_m(p_1,\ldots,p_m)$ is equal to

$$d_m := 2^{2m}(2^{2m-1}-1)\frac{B_m}{(2m)!},$$

see [11, p. 12]. By the signature theorem of Hirzebruch [11, Thm. 8.2.2],

$$\sigma(W_0^{4m}) = d_m p_m[W_0^{4m}]. \tag{2}$$

The formula for $p_m[W_0^{4m}]$ follows by comparing the expressions (1) and (2) for $\sigma(W_0^{4m})$. $\square$

3. Almost complex manifolds

Starting from the manifold W_0^{4m} of the preceding section, we now want to construct almost complex $4m$-manifolds with certain special features. To do this, we shall need to analyse the top-dimensional obstruction to the existence of almost complex structures. Strictly speaking, only the case m even will be relevant for the application to contact geometry, but the construction for m odd is analogous, and of interest because of the following result.

Proposition 4. *The Euler characteristic of an almost complex manifold of (real) dimension $4m$ with first Chern class $c_1 \equiv 0 \bmod 2$ and vanishing decomposable Chern numbers is divisible by*

$$a_m \cdot \text{denominator}\left(\frac{B_m}{2m}\right) \cdot (2m-1)!.$$

There are almost complex manifolds M_0^{4m} with the described properties that realise this minimal (non-zero) absolute value of the Euler characteristic.

Remarks. (1) Such manifolds have been found previously by Puschnigg (unpublished), cf. [12, p. 778]; his diploma thesis [20] contains a homotopy-theoretic proof of (essentially) the above proposition, but no explicit construction of the M_0^{4m}.

(2) If only the vanishing of the decomposable Chern numbers is required, the formula for the minimal positive Euler characteristic is as above, but without the factor a_m, see [5] and [1]. No explicit manifolds realising this value were given in those papers.

Proof. The divisibility statement is contained in [5, Lemma 2.3]. Since the proof there is left to the reader, here is a brief sketch, which is intended merely to indicate what integrality statements enter into the proof. We assume for the time being that the reader is familiar with the notations for various classical genera of manifolds; in the more formal arguments below all these notations will be explained.

Let M be a manifold satisfying the assumptions of the proposition. The condition on c_1 implies that M is a spin manifold, hence $\hat{A}[M] \in a_m\mathbb{Z}$, see [11, p. 198/9]. Because of the vanishing of the decomposable Chern numbers, only the coefficient of the Pontrjagin class p_m in the $\hat{A}$-polynomial is relevant for the computation of the genus $\hat{A}[M]$. The way to compute that coefficient is described in [11, Section 1.4], and one finds

$$\hat{A}[M] = -\frac{1}{(2m-1)!} \cdot \frac{B_m}{4m} \cdot p_m[M].$$

We conclude that

$$a_m \cdot (2m-1)! \cdot \text{denominator}\left(\frac{B_m}{4m}\right) \mid \text{numerator}\left(\frac{B_m}{4m}\right) \cdot p_m[M].$$

From the integrality of $\hat{A}[M]$ and $\langle \mathrm{ph}(M)\hat{A}(M), [M]\rangle$ for the spin manifold M, and the vanishing of the decomposable Chern numbers, one finds with a little computation that

$$(2m-1)! \mid p_m[M].$$

Hence

$$a_m \cdot (2m-1)! \cdot \text{denominator}\left(\frac{B_m}{4m}\right) \mid p_m[M].$$

The factor a_m in that last divisibility statement is justified by the fact that the numerator of $B_m/4m$ is odd; we do not need to use the stronger statement that for the spin manifold M the genus $\langle \mathrm{ph}(M)\hat{A}(M), [M]\rangle$ is an integer divisible by a_m.

Finally, the vanishing of the decomposable Chern numbers implies that $p_m[M]$ is (up to sign) twice the Euler characteristic of M. This yields the claimed divisibility of the Euler characteristic.

We now turn to the construction of the manifolds M_0^{4m}. Let W be an oriented $4m$-dimensional manifold, and assume that there exists an almost complex structure J on $W - D^{4m}$ for some embedded disc D^{4m}. Write

$$\mathfrak{o}(W,J) \in H^{4m}(W; \pi_{4m-1}(\mathrm{SO}_{4m}/\mathrm{U}_{2m})) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z}_2 & \text{for } m \equiv 0 \bmod 2, \\ \mathbb{Z} & \text{for } m \equiv 1 \bmod 2, \end{cases}$$

for the obstruction to extending J as an almost complex structure over W. Here the splitting $\pi_{8k-1}(\mathrm{SO}_{8k}/U_{4k}) \cong \mathbb{Z}\oplus\mathbb{Z}_2$ is defined by identifying the $\mathbb{Z}$-summand with the kernel of the stabilising map

$$\pi_{8k-1}(\mathrm{SO}_{8k}/U_{4k}) \longrightarrow \pi_{8k-1}(\mathrm{SO}/\mathrm{U}) \cong \mathbb{Z}_2,$$

cf. [14, Lemma 8]. For $m \equiv 0 \bmod 2$ we write $\mathfrak{o} = \mathfrak{o}_0 + \mathfrak{o}_2 \in \mathbb{Z}\oplus\mathbb{Z}_2$; for $m \equiv 1 \bmod 2$ we also write $\mathfrak{o} = \mathfrak{o}_0$ in order to simplify notation below.

Then we have the following statements from [14] and [18]:

(i) $\mathfrak{o}(S^{4m}, J)$ is independent of J and will be written as $\mathfrak{o}(S^{4m})$. We have $\mathfrak{o}(S^{8k}) = (1,0)$ and $\mathfrak{o}(S^{8k-4}) = 1$.

(ii) Almost complex structures J on $W - D^{4m}$ and J' on $W' - D^{4m}$ give rise to a natural almost complex structure $J + J'$ on $W \# W' - D^{4m}$ (which coincides with J or J' along the $(4m-1)$-skeleton of W or W', respectively) such that

$$\mathfrak{o}(W\#W', J+J') = \mathfrak{o}(W,J) + \mathfrak{o}(W',J') - \mathfrak{o}(S^{4m}).$$

(iii) Write p_m for the top-dimensional Pontrjagin class of the tangent bundle of W and c_i for the Chern classes of the complex bundle $(T(W - D^{4m}), J)$. Then the obstruction $\mathfrak{o}_0(W, J)$ can be computed explicitly as

$$\mathfrak{o}_0(W, J) = \frac{1}{2}\chi(W) + \frac{1}{4}\langle(-1)^{m+1}p_m + \sum_{\substack{i+j=2m \\ i,j\geq 1}} (-1)^i c_i c_j, [W]\rangle,$$

where $\chi(W)$ denotes the Euler characteristic of W. If J extends over W as a stable almost complex structure $\widetilde{J}$ with top-dimensional Chern class $c_{2m}(\widetilde{J})$, this formula simplifies to

$$\mathfrak{o}_0(W, J) = \frac{1}{2}\left(\chi(W) - \langle c_{2m}(\widetilde{J}), [W]\rangle\right).$$

Since the tangent bundle $T(W_0^{4m} - D^{4m})$ is trivial, the manifold $W_0^{4m} - D^{4m}$ admits an almost complex structure J_0 inducing the natural orientation and with vanishing Chern class c_m. Thus, we obtain

$$\begin{aligned} \mathfrak{o}_0(W_0^{4m}, J_0) &= \frac{1}{2}\,\chi(W_0^{4m}) + \frac{(-1)^{m+1}}{4}\,p_m[W_0^{4m}] \\ &= 1 + \frac{1}{2}\left(1 + \frac{(-1)^{m+1}}{2d_m}\right)\sigma(W_0^{4m}). \end{aligned}$$

(a) For $m = 2k-1$ an odd integer, $q := \mathfrak{o}(W_0^{8k-4}, J_0)$ is a positive integer. The parallelisable manifold $S^1 \times S^{8k-5}$ admits an almost complex structure J', hence $\mathfrak{o}(S^1 \times S^{8k-5}, J') = 0$. Define

$$M_0^{8k-4} = W_0^{8k-4} \#_q (S^1 \times S^{8k-5}).$$

By (i) and (ii), the manifold M_0^{8k-4} admits an almost complex structure. The Euler characteristic of this manifold is given by

$$\begin{aligned} \chi(M_0^{8k-4}) &= 2 + \sigma(W_0^{8k-4}) - 2q \\ &= -\frac{1}{2} \cdot p_{2k-1}[W_0^{8k-4}] \\ &= -2 \cdot \text{denominator}\left(\frac{B_{2k-1}}{2(2k-1)}\right) \cdot (2(2k-1)-1)!. \end{aligned}$$

(b) For $m = 2k$ an even integer, we want to show that $q := \mathfrak{o}_0(W_0^{8k}, J_0)$ is a negative integer. A famous formula of Euler, cf. [19, p. 286], states that

$$\frac{B_m(2\pi)^{2m}}{2(2m)!} = \sum_{\nu=1}^{\infty} \frac{1}{\nu^{2m}}.$$

Hence, with d_m as defined in the proof of Lemma 3, we get (with $m \geq 2$)

$$\begin{aligned} d_m &= \frac{2(2^{2m-1}-1)}{\pi^{2m}} \sum_{\nu=1}^{\infty} \frac{1}{\nu^{2m}} \\ &\leq \frac{2(2^{2m-1}-1)}{\pi^{2m}} \sum_{\nu=1}^{\infty} \frac{1}{\nu^{2}} \\ &= \frac{2(2^{2m-1}-1)}{\pi^{2m}} \cdot \frac{\pi^2}{6} \\ &\leq \left(\frac{2}{\pi}\right)^{2m} \cdot 2 \;\leq\; \left(\frac{2}{3}\right)^{4} \cdot 2 \;=\; \frac{32}{81}. \end{aligned}$$

It follows that

$$\frac{1}{2} - \frac{1}{4d_{2k}} \leq \frac{1}{2} - \frac{81}{128} = -\frac{17}{128}$$

and

$$q = 1 + \left(\frac{1}{2} - \frac{1}{4d_{2k}}\right)\sigma(W_0^{8k}) \leq 1 - \frac{17}{128} \cdot 8 < 0.$$

Define

$$M_0^{8k} = W_0^{8k} \#_{-q} (S^{4k} \times S^{4k}).$$

Lemma 5. *The manifold M_0^{8k} admits an almost complex structure.*

Proof. Since the manifold $S^{4k} \times S^{4k}$ has stably trivial tangent bundle, we can find an almost complex structure J' on $S^{4k} \times S^{4k} - D^{8k}$ that extends as a stable structure with vanishing Chern classes. Hence $\mathfrak{o}_0(S^{4k} \times S^{4k}, J') = 2$ by (iii) and $\mathfrak{o}_0(M_0^{8k}, J_0 + (-q)J') = 0$ by (i) and (ii).

The vanishing of $\mathfrak{o}_2$ can be deduced from [9, Thm. 1.2]. This theorem states that subject to the condition $\mathrm{Sq}^2 H^{8k-2}(M_0^{8k}) = 0$, which is obviously satisfied, we have $\mathfrak{o}_2 = 0$ if and only if

$$\langle \mathrm{ph}(M_0^{8k}) \hat{A}(M_0^{8k}), [M_0^{8k}] \rangle \equiv 0 \bmod 2. \tag{3}$$

Here $\mathrm{ph}(M_0^{8k})$ denotes the Pontrjagin character of TM_0^{8k}, i.e. the Chern character of $TM_0^{8k} \otimes \mathbb{C}$, and $\hat{A}(M_0^{8k}) = \sum_{\nu=0}^{\infty} \hat{A}_\nu(p_1, \ldots, p_\nu)$ the full $\hat{A}$-polynomial in the Pontrjagin classes of M_0^{8k}. Write

$$\mathrm{ph}(M_0^{8k}) = 8k + \frac{s_{2k}}{(2k)!} + \frac{s_{4k}}{(4k)!}$$

with $s_j \in H^{2j}(M_0^{8k})$, cf. [11, p. 92]. From the Newton formulae

$$s_j - c_1 s_{j-1} + \ldots + (-1)^j c_j \cdot j = 0,$$

where the c_i are the Chern classes of $TM_0^{8k} \otimes \mathbb{C}$, we find $s_{2k} = -2k \cdot c_{2k} = (-1)^{k+1} 2k \cdot p_k(M_0^{8k}) = 0$ and $s_{4k} = -4k \cdot c_{4k} = -4k \cdot p_{2k}(M_0^{8k})$. Hence

$$\mathrm{ph}(M_0^{8k}) = 8k - \frac{p_{2k}(M_0^{8k})}{(4k-1)!},$$

and therefore

$$\langle \mathrm{ph}(M_0^{8k})\hat{A}(M_0^{8k}), [M_0^{8k}]\rangle = 8k \cdot \hat{A}[M_0^{8k}] - \frac{p_{2k}[M_0^{8k}]}{(4k-1)!}.$$

The first summand is even, since $\hat{A}[M_0^{8k}]$ is an integer. By the proof of Lemma 3 — notice that $\sigma(M_0^{8k}) = \sigma(W_0^{8k})$ —, the second summand equals the denominator of $B_{2k}/8k$, which is even, and thus indeed $\mathfrak{o}_2 = 0$ by the cited result of [9]. □

Remark. Concerning the vanishing of the stable obstruction $\mathfrak{o}_2$, Puschnigg [21] has shown us a more general K-theoretic argument for the existence of a stable almost complex structure on every almost parallelisable manifold of even (real) dimension.

The Euler characteristic of M_0^{8k} equals

$$\begin{aligned} \chi(M_0^{8k}) &= 2 + \sigma(W_0^{8k}) - 2q \\ &= \frac{1}{2} \cdot p_{2k}[W_0^{8k}] \\ &= \text{denominator}\left(\frac{B_{2k}}{2 \cdot 2k}\right) \cdot (2 \cdot 2k - 1)!. \end{aligned}$$

This concludes the proof of Proposition 4. □

4. Proof of Theorems 1 and 2

Theorem 1 for spheres of dimension $\equiv 7 \bmod 8$ follows from the existence of the $(4k-1)$-connected almost complex $8k$-manifold M_0^{8k} via [8, Thm. 24]. Given what was known previously, this completes the proof of Theorem 1.

Theorem 2 follows similarly. Write J for the almost complex structure on M_0^{8k}. Then obviously $\mathfrak{o}(M_0^{8k}, J) = 0$. By [14], the manifold $-M_0^{8k}$ (that is, M_0^{8k} with reversed orientation) admits, on the complement of a disc, an almost complex structure J' with

$$\mathfrak{o}(-M_0^{8k}, J') = -\mathfrak{o}(M_0^{8k}, J) + \chi(M_0^{8k})\mathfrak{o}(S^{8k}) = \chi(M_0^{8k}) \cdot (1, 0).$$

(For the nonstable part $\mathfrak{o}_0$ of the obstruction one can also deduce this from the formula given in (iii) above.) Thus, for a and b non-negative integers, the $(4k-1)$-connected manifold $\#_a M_0^{8k} \#_b (-M_0^{8k}) - D^{8k}$ admits an almost complex structure whose nonstable part of the obstruction to extension over the closed manifold, by (i) and (ii), is equal to

$$\mathfrak{o}_0 = a \cdot 0 + b\chi(M_0^{8k}) - (a + b - 1) = b\big(\chi(M_0^{8k}) - 1\big) - (a - 1).$$

By a suitable choice of a and b, any integer can be realised. (For spheres of dimension $8k+3$, use the $(4k+1)$-connected almost complex $(8k+4)$-manifold constructed in [8, Thm. 25].) The case of S^7 is special since $\mathbb{H}P^2 - D^8$ admits

an almost complex structure with $\mathfrak{o} = (0,1) \in \mathbb{Z} \oplus \mathbb{Z}_2$, which allows us also to vary the stable part of the obstruction. (The obstruction class $\mathfrak{o}$ for $\mathbb{H}P^2$ was computed in [8]; instead of the argument employed there one may use the result of [9] quoted above.) This concludes the proof of Theorems 1 and 2.

We end this note with a brief discussion of the failure of our methods to cover stably nontrivial homotopy classes of almost contact structures on spheres of dimension $\equiv 7 \bmod 8$ (and greater than 7). The key to this is the observation that equation (3) (for M^{8k}), which contains purely topological data, is both necessary and sufficient for *any* almost complex structure on $M^{8k} - D^{8k}$ to extend as a stable structure over M^{8k} (this follows from the proof of [9, Thm. 1.2]). Thus, in order to construct an exotic contact structure in a stably nontrivial homotopy class of almost contact structures, our methods would require us to find a $(4k-1)$-connected $8k$-manifold M^{8k} admitting an almost complex structure on the complement of a disc and satisfying

$$\langle \mathrm{ph}(M^{8k})\hat{A}(M^{8k}), [M^{8k}]\rangle \equiv 1 \bmod 2.$$

There are two (not entirely unrelated) constructions of such highly connected manifolds: the Brieskorn construction described in [8, Section 5.1] and the plumbing of (multiples of) the tangent disc bundle of S^{4k}. Both give rise to *parallelisable* $8k$-manifolds with boundary (for the former this is obvious; for the latter see [13, Satz 8.7]). Therefore, Puschnigg's observation shows that neither of these constructions yields the desired contact structures. In part (a) of the following proposition we give an independent proof of this observation for the dimensions of interest to us. In part (b) we show that in dimension 15, at least, the construction via highly connected almost complex manifolds fails altogether.

Proposition 6. *(a) If M^{8k} is an almost parallelisable manifold, then the genus $\langle \mathrm{ph}(M^{8k})\hat{A}(M^{8k}), [M^{8k}]\rangle$ is even. Hence M^{8k} admits a stable almost complex structure.*

(b) If M^{16} is a spin manifold whose Pontrjagin numbers other than p_2^2 and p_4 vanish, then $\langle \mathrm{ph}(M^{16})\hat{A}(M^{16}), [M^{16}]\rangle$ is even.

Proof. (a) Under the assumptions of the proposition, the only (potentially) nonvanishing Pontrjagin class of M^{8k} is $p_{2k}(M^{8k})$. Hence, like in the proof of Lemma 5 we find

$$\langle \mathrm{ph}(M^{8k})\hat{A}(M^{8k}), [M^{8k}]\rangle = 8k \cdot \hat{A}[M^{8k}] - \frac{p_{2k}[M^{8k}]}{(4k-1)!}.$$

By an argument similar to the proof of the divisibility statement in Proposition 4, $p_{2k}[M^{8k}]$ is divisible by denominator$(B_{2k}/8k) \cdot (4k-1)!$, cf. [12,

p. 770]. Since the denominator of every Bernoulli number is even, the claim on the parity of the genus follows. As an almost parallelisable manifold, M^{8k} obviously admits an almost complex structure on the complement of a disc, which extends as a stable almost complex structure over the closed manifold by [9, Thm. 1.2]. (The condition $\mathrm{Sq}^2 H^{8k-2}(M^{8k}) = 0$ of that theorem is satisfied because of the almost parallelisability of M^{8k}.)

(b) By computations analogous to the proof of Lemma 5 and using the formula (modulo terms involving p_1 or p_3)

$$\hat{A}(M^{16}) = 1 - \frac{1}{2^5 \cdot 3^2 \cdot 5}\, p_2(M^{16}) + \frac{1}{2^{16} \cdot 3^4 \cdot 5^2 \cdot 7}\big(416\, p_2^2(M^{16}) - 384\, p_4(M^{16})\big)$$

one finds

$$\langle \mathrm{ph}(M^{16})\hat{A}(M^{16}), [M^{16}]\rangle = 496\, \hat{A}[M^{16}],$$

which is even because of the integrality of the $\hat{A}$-genus for spin manifolds. □

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